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ON DYNAMIC STRESS CONCENTRATIONS IN ELASTIC PLATES

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Abstract—Use is made of an approximate two dimensional plate theory, which takes into account some aspects of the plate thickness, to study the reflection of an infinite train of extensional waves from a circular traction free boundary. The results are investigated to see under what conditions these results can be brought into analytic agreement with those predicted by generalized plane stress theory when the latter is used to solve the same problem.

INTRODUCTION

THE effects produced by a small discontinuity on the stress field in a structural member which is subjected to a static loading condition constitute an obviously important engineering problem and have been the object of numerous analytical and experimental studies. The effects produced when the loading is dynamic, while no less important, have received far less attention. Recently, however, several authors have considered this problem and in one such investigation Pao [1] used generalized plane stress theory to study the effects produced by a circular hole in an infinite elastic plate on an infinite train of extensional waves. His motivation was to study the concentration of stress in the immediate vicinity of the hole.

Pao's solution represents an approximation, however, inasmuch as the generalized plane stress theory takes into account only average displacements and stresses through the thickness of the plate. Experience gained by using this theory to investigate various problems concerned with the steady vibrations of plates or the propagation of waves in plates has shown it to be valid so long as the plate is sufficiently thin and the frequency is sufficiently low. Nevertheless, in the present case, a doubt remains as to the validity of the approximation since the problem obviously involves a localized phenomenon.

It was the purpose of the present investigation to obtain some information on the validity of using generalized plane stress theory to study such problems. Due to mathematical complexities, it is not feasible to conduct such a study within the framework of the three dimensional equations of elasticity, however, there are available second order equations [2] which are intermediate between the generalized plane stress theory and the full linear theory and which are solvable.

Specifically, the report considers the reflection of an infinite train of extensional waves from a circular traction free boundary. This wave is taken to propagate in the lowest of the three extensional modes of propagation that are contained in the second order equations. It is only this lowest mode which has a counterpart in the generalized plane stress theory. Upon striking the boundary, the incident wave must generate waves from all three extensional modes of propagation if the traction free condition is to be maintained. In addition, since the incident wave is not taken, in general, to be axially symmetric, the two face shearing modes of propagation that are contained in the second order theory will also be generated. The amplitude ratios of the five reflected waves with respect to the incident wave are obtained and investigated to determine under what conditions the results will agree with those that would be obtained if generalized plane stress theory was used to solve the same problem. To answer this requires that a definition of what is meant by agreement be given. Of the five waves predicted by the second order equations only the lowest extensional wave and the lowest face shear wave have counterparts in the generalized plane stress equations. Furthermore, since the propagation constants of the remaining three waves are complex at low frequencies, they will offer contributions to the solution which will decay exponentially with distance from the boundary. For these reasons, the solutions will be said to agree when they predict results for the lowest extensional wave and the lowest face shear wave which agree.

Briefly, the results show that the second order solution "reduces" to the generalized plane stress solution as the non-dimensionalized product $\omega b/(\mu/\rho)^{\frac{1}{2}}$ (i.e. ω being the circular frequency of the incident wave, b being the half plate thickness and μ and ρ being material constants) goes to zero provided the ratio b/a (i.e. a being the radius of the hole) also approaches zero. If, however, the ratio b/a is finite then the two solutions will differ even in the limit of $\omega b/(\mu/\rho)^{\frac{1}{2}}$ equalling zero.

This result may lead one to conclude that the generalized plane stress approximation will be valid at low frequencies provided the radius of the cavity is several times larger than the plate thickness. For cavities of size comparable to the plate thickness, however, the generalized plane stress solution must be suspect. It might be pointed out that similar doubts must also be cast on the second order solution although one might hope that it would give a truer picture for a slightly smaller cavity.

Finally, it might be remarked that similar questions could be raised concerning the use of classical elasticity theory to investigate the effects of an infinitely long cylindrical cavity on the stress field in an infinite elastic medium. These questions arise since elasticity theory neglects the structure which is present on the elemental level of all polycrystalline materials. In seeking to answer these questions for static loading, several investigations [3, 4] have used continuum theories which take into account some aspects of the geometry of the elemental volume. Their results lead to conclusions which are similar to those in this report if the plate thickness is replaced by a material constant which has the dimensions of length and has been reasoned to be a characteristic dimension of a representative crystal. That is, the elasticity solution agrees with the solution obtained from the higher order theory provided the radius of the cavity is several times larger than a characteristic dimension of the elemental volume. In such cases the elasticity solution represents a valid approximation, whereas, if the cavity is of size comparable to the elemental volume it does not. In the latter case the solutions obtained from the higher order theories must also be considered questionable.

SECOND ORDER EQUATIONS

Consider a thin plate with the upper and lower faces free from traction and let the median surface of the plate be a coordinate plane. The displacement equations of motion governing this plate may be expressed, according to the Mindlin–Medick approximation, in the following form which is invariant under a transformation in the plane of the plate:

$$
\begin{aligned}
\mu\nabla^2\mathbf{u}^{(0)}+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}^{(0)}+\frac{\lambda k_1}{b}\nabla u_2^{(1)} &= \rho\frac{\partial^2 u^{(0)}}{\partial t^2}\\
\mu k_2^2\nabla^2 u_2^{(1)}-\frac{3\lambda k_1}{b}\nabla\cdot\mathbf{u}^{(0)}-\frac{3(\lambda+2\mu)k_1^2}{b^2}u_2^{(1)}+\frac{3\mu k_2^2}{b}\nabla\cdot\mathbf{u}^{(2)} &= \rho k_3^2\frac{\partial^2 u_2^{(1)}}{\partial t^2}\\
\frac{E'}{2}[(1-\nu)\nabla^2\mathbf{u}^{(2)}+(1+\nu)\nabla\nabla\cdot\mathbf{u}^{(2)}]-\frac{5\mu k_2^2}{b}\left(\nabla u_2^{(1)}+\frac{3}{b}\mathbf{u}^{(2)}\right) &= \rho k_4^2\frac{\partial^2\mathbf{u}^{(2)}}{\partial t^2}
\end{aligned}
\tag{1}
$$

where

$$
E' = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}. \tag{2}
$$

In these equations, b is the half-plate thickness, λ and μ are Lamé's constants, ρ is the mass density, ν is Poisson's ratio and k_i ($i = 1, \ldots, 4$) are correction factors which are introduced in order to improve the agreement between the infinite plate frequency spectrum predicted by these equations and that predicted by the full three dimensional theory. In addition, ∇ is the two dimensional gradient operator while $\mathbf{u}^{(0)}$, $u_2^{(1)}$, and $\mathbf{u}^{(2)}$ represent the displacement field allowed by these equations. The vectors $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(2)}$ lie in the plane of the plate, the former representing a displacement which is uniform across the thickness and the latter representing a displacement which varies quadratically across the thickness. The scalar $u_2^{(1)}$ represents a displacement normal to the plane of the plate which varies linearly across the thickness.

The state of stress at a point may be invariantly expressed by means of two dyadics, $\underline{\underline{\tau}}^{(0)}$ and $\underline{\underline{\tau}}^{(2)}$, one vector, $\underline{\tau}_2^{(1)}$, and one scalar $\tau_{22}^{(0)}$. The physical interpretation to be given to

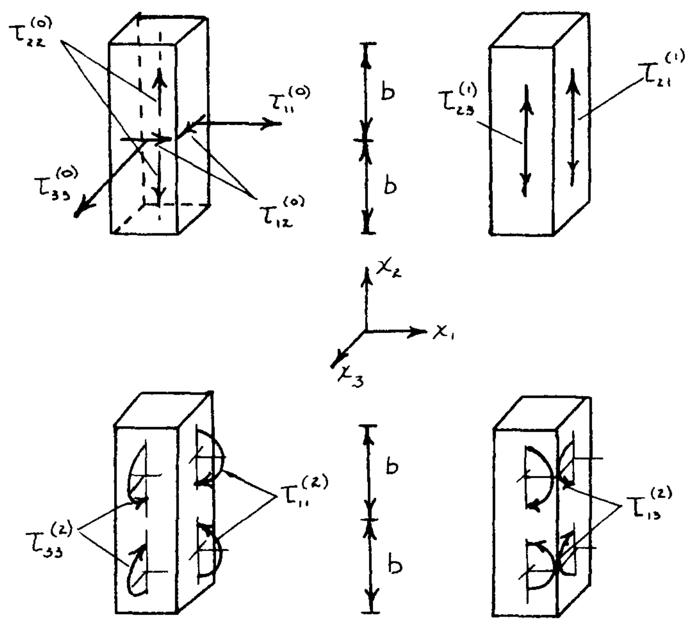

FIG. 1. Stress resultants referred to Cartesian coordinate system.

these quantities is shown, for a Cartesian coordination system, in Fig. 1. These stresses are related to the displacements according to

$$\begin{aligned}
\underline{\underline{\tau}}^{(0)} &= 2\left[\lambda\left(\nabla\cdot\mathbf{u}^{(0)}+\frac{k_1}{b}u_2^{(1)}\right)\underline{\underline{I}}+\mu\left(\nabla\mathbf{u}^{(0)}+\mathbf{u}^{(0)}\nabla\right)\right] \\
\underline{\underline{\tau}}^{(2)} &= \tfrac{2}{5}[E'\nu(\nabla\cdot\mathbf{u}^{(2)})\underline{\underline{I}}+\mu(\nabla\mathbf{u}^{(2)}+\mathbf{u}^{(2)}\nabla)] \\
\underline{\tau}_2^{(1)} &= \frac{2\mu k_2^2}{b}\mathbf{u}^{(2)}+\frac{2\mu k_2^2}{3}\nabla u_2^{(1)} \\
\tau_{22}^{(0)} &= 2\left[\frac{(\lambda+2\mu)k_1^2}{b}u_2^{(1)}+\lambda k_1\nabla\cdot\mathbf{u}^{(0)}\right]
\end{aligned} \tag{3}$$

In the case of displacements varying harmonically with time it is convenient to introduce displacement potential functions ϕ_j $(j = 1, 2, 3)$ and $\boldsymbol{\psi}_j = \psi_j\mathbf{k}$ $(j = 1, 2)$, where $\mathbf{k}$ represents a unit vector normal to the median plane of the plate. The displacements are expressed in terms of these potential functions as follows:

$$\begin{aligned}
\mathbf{u}^{(0)} &= \sum_{j=1}^{3}\nabla\phi_j+\nabla\times\boldsymbol{\psi}_1 \\
u_2^{(1)} &= \sum_{j=1}^{3}\alpha_j\phi_j \\
\mathbf{u}^{(2)} &= \sum_{j=1}^{3}\beta_j\nabla\phi_j+\nabla\times\boldsymbol{\psi}_2,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\alpha_j &= \frac{b}{\lambda k_1}[(\lambda+2\mu)\xi_j^2-\rho\omega^2] \qquad j = 1, 2, 3 \\
\beta_j &= \frac{\alpha_j b}{\left[\frac{3}{4}(\Omega^2-4)-\dfrac{E'b}{5\mu k_2^2}\xi_j^2\right]} \qquad j = 1, 2, 3
\end{aligned} \tag{5}$$

In equations (5) ω is the circular frequency and

$$\Omega = \frac{\omega}{\omega_s}; \qquad \omega_s = \frac{\pi}{2b}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \tag{6}$$

In addition ξ_j $(j = 1, 2, 3)$ are the propagation constants of the three extensional modes and are given by the cubic equation, which is expressed in the following determinantal form,

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{vmatrix} = 0, \tag{7}$$

where

$$
\begin{aligned}
a_{11} &= K^2Z^2-\Omega^2 \\
a_{22} &= \frac{k_2^2}{3}Z^2+\frac{4K^2k_1^2}{\pi^2}-\frac{k_3^2}{3}\Omega^2 \\
a_{33} &= \frac{E'}{3}Z^2+\frac{12k_2^2}{\pi^2}-\frac{k_4^2}{5}\Omega^2 \\
a_{12} &= \frac{2(K^2-2)k_1}{\pi}Z \\
a_{23} &= -\frac{2k_2^2}{\pi}Z
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
Z &= \frac{2\xi b}{\pi} \\
K^2 &= \frac{\lambda+2\mu}{\mu} = \frac{2(1-\nu)}{(1-2\nu)}
\end{aligned}
\tag{9}
$$

The potentials ϕ_j, ψ_1 and ψ_2 satisfy Helmholtz equations:

$$
\begin{aligned}
\nabla^2\phi_j+\xi_j^2\phi_j &= 0 \qquad j = 1, 2, 3 \\
\mu\nabla^2\psi_1+\rho\omega^2\psi_1 &= 0 \\
\mu\nabla^2\psi_2+\left(\rho k_4^2\omega^2-\frac{15\mu k_2^2}{b^2}\right)\psi_2 &= 0
\end{aligned}
\tag{10}
$$

INCIDENT AND REFLECTED WAVES

Consider the case of an infinite train of waves which propagate in the lowest extensional mode towards the origin in an infinite elastic plate. Such a wave train can be represented by a potential ϕ_1 of the form:

$$\phi_1^{(i)}A_0H_n^{(2)}(\xi_1 r)\cos n\theta \exp(-i\omega t) \tag{11}$$

where $H_n^{(2)}(x)$ represents a Hankel function of the second kind of order n. Equation (11) satisfies the appropriate Helmholtz equation provided ξ_1 represents the propagation constant of the lowest extensional mode.

Upon striking a traction free boundary at, $r = a$, the incident wave will generate outgoing waves of the form:

$$
\begin{aligned}
\phi_j^{(r)} &= A_jH_n^{(1)}(\xi_j r)\cos n\theta \exp(-i\omega t); \qquad j = 1, 2, 3 \\
\psi_j^{(r)} &= B_jH_n^{(1)}(\delta_j r)\sin n\theta \exp(-i\omega t); \qquad j = 1, 2
\end{aligned}
\tag{12}
$$

where $H_n^{(1)}(x)$ represents a Hankel function of the first kind of order n. The potentials defined by equations (12) will satisfy equations (10) provided the ξ_j's represent the propagation constants of the extensional modes and the δ_j's represent the propagation constants of the face shear modes given by

$$\delta_1^2 = \frac{\rho\omega^2}{\mu} \tag{13}$$

and

$$\delta_2^2 = \frac{\rho k_4^2\omega^2}{\mu} - \frac{15k_2^2}{b^2}$$

The amplitudes of the reflected waves are determined by the condition to be imposed on the boundary. For the case at hand, the appropriate condition is that the sum of the tractions arising from the incoming wave and outgoing waves must vanish on $r = a$. This condition will be insured provided

$$\tau_{rr}^{(0)}(r = a) = \tau_{r\theta}^{(0)}(r = a) = \tau_{2r}^{(1)}(r = a)\tau_{rr}^{(2)}(r = a) = \tau_{r\theta}^{(2)}(r = a) = 0 \tag{14}$$

where, in the above set of equations, the τ's represent the sums of stresses.

Substitution of the assumed solutions into equations (4) and the results into equations (3) and then making use of equations (14) results in a system of five linear algebraic equations on the five unknown amplitude ratios ($C_j = A_j/A_0$; $j = 1, 2, 3$ and $C_j = B_{j-3}/A_0$; $j = 4, 5$). Introducing suitable non-dimensional quantities, this system of equations may be represented in the following form:

$$T_{ij}C_j = -T_{i1}^* \qquad i = 1, 2, 3, 4, 5 \tag{15}$$

where the repeated index indicates summation over the range one to five and the T_{ij}'s are given by

$$\begin{aligned}
T_{ij} &= \left[\frac{2n(n-1)}{S^2} - \Omega^2\right] H_n^{(1)}(Z_jS) + \frac{2}{S}Z_jH_{n+1}^{(1)}(Z_jS) \qquad j = 1, 2, 3\\
T_{2j} &= \frac{2n}{S}\left[Z_jH_{n+1}^{(1)}(Z_jS) - \frac{n-1}{S}H_n^{(1)}(Z_jS)\right] \qquad j = 1, 2, 3\\
T_{3j} &= \frac{\beta_j}{1-\nu}\left\{\left[\frac{(1-\nu)n(n-1)}{S^2} - Z_j^2\right] H_n^{(1)}(Z_jS) + \frac{1-\nu}{S}Z_jH_{n+1}^{(1)}(Z_jS)\right\} \qquad j = 1, 2, 3\\
T_{4j} &= \beta_jT_{2j} \qquad j = 1, 2, 3\\
T_{5j} &= \left(\beta_j + \frac{\bar{\alpha}_j}{3}\right)\left[\frac{n}{S}H_n^{(1)}(Z_jS) - Z_jH_{n+1}^{(1)}(Z_jS)\right] \qquad j = 1, 2, 3\\
T_{14} &= \frac{2n}{S}\left[Z_4H_{n+1}^{(1)}(Z_4S) - \frac{n-1}{S}H_n^{(1)}(Z_4S)\right]\\
T_{24} &= \left[\frac{2n(n-1)}{S^2} - Z_4^2\right] H_n^{(1)}(Z_4S) + \frac{2}{S}Z_4H_{n+1}^{(1)}(Z_4S)\\
T_{35} &= \frac{n}{S}\left[Z_5H_n^{(1)}(Z_5S) - \frac{n-1}{S}H_n^{(1)}(Z_5S)\right]\\
T_{45} &= \left[\frac{2n(n-1)}{S^2} - Z_5^2\right] H_n^{(1)}(Z_5S) + \frac{2}{S}Z_5H_{n+1}^{(1)}(Z_5S)
\end{aligned} \tag{16}$$

$$T_{55} = \frac{n}{S} H_n^{(1)}(Z_5 S)$$

$$T_{15} = T_{25} = T_{34} = T_{44} = T_{54} = 0$$

In addition in equations (15) T_{i1}^* indicates the complex conjugate of T_{i1} and in equations (16) the following non-dimensional terms have been introduced, in addition to those already given in equations (16) and equations (9).

$$\begin{aligned} Z_{4,5} &= 2b\delta_{1,2}/\pi \\ \bar{\alpha}_j &= b\alpha_j \\ S &= \pi a/2b \end{aligned} \tag{17}$$

The corresponding generalized plane stress solution consists of the first two algebraic equations of the set represented by equations (15) if C_2 and C_3 are set equal to zero and Z_1 is taken as its limiting form as Ω goes to zero. It is not difficult to show that the generalized plane stress solution is not dependent on the ratio $S = \pi a/2b$.

ASYMPTOTIC SOLUTIONS

It remains to investigate whether the solution for C_1 and C_4, as predicted by equations (15), reduces to the corresponding generalized plane stress solution as the limit of the non-dimensionalized product $\Omega = 2\omega b/\pi(\mu/\rho)^{\frac{1}{2}}$ approaches zero. Since, as already stated, the first two algebraic equations represent the generalized plane stress solution in the limit of Ω approaching zero provided C_2 and C_3 are zero, the reduction will be accomplished if the contributions of the higher modes to these equations vanish in the limit. It shall be seen that whether or not this is the case depends on the ratio $S = \pi a/2b$, and that we can differentiate between several distinct cases.

In the first case, S will be taken to approach infinity at the same rate as Ω approaches zero. This limiting case may be interpreted as a wave of finite frequency reflecting from a finite hole in an infinitesimally thin plate. In the second case, S will be taken to approach infinity at a slower rate than Ω approaches zero. This limit may be interpreted to correspond to a wave of finite frequency reflecting from an infinitesimal hole in a plate of infinitesimal thickness with the thickness infinitesimal being of higher order. In the third case S will be taken to remain finite and Ω will be taken to approach zero. This situation may be interpreted as in the second case with the order of the two infinitesimals taken as the same. Finally the case is considered in which S is taken to approach zero along with Ω. This case corresponds to cases two and three, only now the order of the infinitesimal representing the hole diameter is higher than the infinitesimal representing the plate thickness.

Case 1. In the first case we wish to investigate the reduction of the above formulation in the double limit of Ω approaching zero and S approaching infinity when the rates of the two approaches are taken to be equal. Introducing the notation $f(x)$ is $O[g(x)]$ to denote $f(x) \leq Mg(x)$, M being some constant, the behavior of S in the vicinity of the desired limit may be expressed by

$$S \text{ is } O(\Omega^{-1}) \tag{18}$$

and the problem of a double limit may be treated as one of a single limit of Ω approaching zero.

Using the definitions of the relevant non-dimensional quantities, they may readily be shown to behave in the limit of Ω approaching zero according to:

$$\begin{aligned} &Z_1 \text{ and } Z_4 \text{ are } O(\Omega) \\ &Z_2, Z_3 \text{ and } Z_5 \text{ are } O(1) \end{aligned} \tag{19}$$

Furthermore,

$$\begin{aligned} &\bar{\alpha}_1 \text{ and } \beta_1 \text{ are } O(\Omega^2) \\ &Z_1S, Z_4S, \bar{\alpha}_j \text{ and } \beta_j \; (j = 2, 3) \text{ are } O(1) \\ &Z_2S, Z_3S \text{ and } Z_5S \text{ are } O(\Omega^{-1}) \end{aligned} \tag{20}$$

From equations (20), it is seen to be permissible to expand the Hankel functions with arguments Z_2S, Z_3S, Z_5S asymptotically for large argument according to the expression

$$H_n^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \exp\left[i\left(x - \frac{2n+1}{4}\pi\right)\right] \tag{21}$$

Performing this expansion results in simplified exp'essions for some of the T_{ij}'s which are given below together with the order of their magni ıdes relative to Ω.

$$\begin{aligned}
T_{1j} &\sim \frac{2Z_j}{S}\left(\frac{2}{\pi Z_jS}\right)^{\frac{1}{2}} \exp\left[i\left(Z_jS - \frac{2n+3}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \qquad j = 2, 3 \\
T_{2j} &\sim \frac{2n}{S} Z_j\left(\frac{2}{\pi Z_jS}\right)^{\frac{1}{2}} \exp\left[i\left(Z_jS - \frac{2n+3}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \qquad j = 2, 3 \\
T_{3j} &\sim \frac{\beta_j}{1-\nu} Z_j^2\left(\frac{2}{\pi Z_jS}\right)^{\frac{1}{2}} \exp\left[i\left(Z_jS - \frac{2n+1}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \qquad j = 2, 3 \\
T_{4j} &\sim \beta_j T_{2j} = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \qquad j = 2, 3 \\
T_{5j} &\sim -\left(\beta_j + \frac{\bar{\alpha}_j}{3}\right) Z_j\left(\frac{2}{\pi Z_jS}\right)^{\frac{1}{2}} \exp\left[i\left(Z_jS - \frac{2n+3}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \qquad j = 2, 3 \\
T_{35} &\sim -\frac{n}{S} Z_s\left(\frac{2}{\pi Z_5S}\right)^{\frac{1}{2}} \exp\left[i\left(Z_5S - \frac{2n+3}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \\
T_{45} &\sim -Z_5^2\left(\frac{2}{\pi Z_5S}\right)^{\frac{1}{2}} \exp\left[i\left(Z_5S - \frac{2n+1}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})] \\
T_{55} &\sim \frac{n}{S}\left(\frac{2}{\pi Z_4S}\right)^{\frac{1}{2}} \exp\left[i\left(Z_5S - \frac{2n+1}{4}\pi\right)\right] = O[\Omega^{\frac{1}{2}} \exp(-N\Omega^{-1})]
\end{aligned} \tag{22}$$

In the above, N represents some constant. The forms of the remaining expressions are not simplified but to them can be attached the following order of magnitudes.

$$\begin{aligned} &T_{i1} \text{ and } T_{i4} \text{ are } O(\Omega^2); \qquad i = 1, 2 \\ &T_{i1} \text{ is } O(\Omega^4); \qquad i = 3, 4 \\ &T_{51} \text{ is } O(\Omega^3) \end{aligned} \tag{23}$$

Introduction of the notation $C'_j = C_j \exp(iZ_jS)$ and investigation of the last three equations shows that unless $C'_j \leq O(\Omega^{\frac{7}{2}})$; $j = 2, 3$, the expressions containing C'_j will contribute terms with greater order of magnitudes than the inhomogeneity of the equation. Taking $C'_j = O(\Omega^{\frac{7}{2}})$, it may be seen that this offers a contribution to the first two equations of $O(\Omega^5)$. The remaining terms in the first two equations, however, are all of magnitude $O(\Omega^2)$ and hence if only the terms of the largest order of magnitude are considered, there will be no contribution from the higher modes. Of course, it is necessary to insure that the first two equations are sufficient to determine C_1 and C_4 considering only terms to this order of magnitude; that is, we must show that the first two equations are linearly independent to this order of magnitude. As long as S is $O(\Omega^{-1})$ it is possible to show that this is the case.

Thus we see that the generalized plane stress theory and the Mindlin–Medick theory will predict identical results for the propagating extensional and shear modes in the limit of Ω approaching zero provided the ratio a/b is taken to approach infinity at the same rate.

Case 2. In the second case we wish to investigate the reduction in the double limit of Ω approaching zero and S approaching infinity when the rates of the two approaches are not equal. This can be accomplished by stipulating that S is $O(\Omega^{p-1})$ with $0 < p < 1$ and treating the problem of the double limit as a single limit problem of Ω approaching zero.

Proceeding as before, this introduces the following changes in the orders of magnitude given in equations (20).

$$
\begin{aligned}
&Z_1S \text{ and } Z_4S \text{ are } O(\Omega^p)\\
&Z_2S \text{ and } Z_3S \text{ and } Z_5S \text{ are } O(\Omega^{p-1})
\end{aligned}
\tag{24}
$$

Equations (24) indicate that so long as $p < 1$ the expansions for the Hankel functions with arguments Z_2S, Z_3S and Z_5S that were given before are still valid and the simplified expressions given in equations (22) are still correct. The order of magnitudes of the various terms change, however, with the new magnitudes being

$$
\begin{aligned}
&T_{ij}\ (i = 1,2,4;\ j = 2,3) \text{ and } T_{i5}\ (i = 3,5) \text{ are } O[\Omega^{3(1-p)/2}\exp(N\Omega^{p-1})]\\
&T_{ij}\ (i = 3,5;\ j = 2,3) \text{ and } T_{45} \text{ are } O[\Omega^{(1-p)/2}\exp(N\Omega^{p-1})].
\end{aligned}
\tag{25}
$$

The size of Z_1S and Z_4S for large Ω allows the small argument expansion of the appropriate Hankel functions. The first term of this expansion is given by

$$
H_n^{(1)}(x) = -i\frac{2^n(n-1)!}{\pi x^n} + O(x^{-n+2}); \qquad x < 1 \tag{26}
$$

Introducing this expansion results in the following simplifications.

$$
\begin{aligned}
&T_{i1} \sim -i\frac{2^{n+1}(n+1)!}{\pi} Z_1^{-n} S^{-(n+2)} = O[\Omega^{-np+2(1-p)}]; \qquad i = 1,2\\
&T_{31} \sim \tfrac{1}{2}\beta_1 T_{i1}\ (i = 1 \text{ or } 2) = O[\Omega^{-np+2(2-p)}]\\
&T_{41} \sim \beta_1 T_{21} = O[\Omega^{-np+2(2-p)}]\\
&T_{51} \sim i\left(\beta_1 + \frac{\bar{\alpha}_1}{3}\right)\frac{2n!}{\pi} Z_1^{-n} S^{-(n+1)} = O[\Omega^{-np+(3-p)}]
\end{aligned}
\tag{27}
$$

Similarly

$$T_{i4} \sim -i\frac{2^{n+1}(n+1)!}{\pi}Z_4^{-n}S^{-(n+2)} = O[\Omega^{-np+2(1-p)}]; \qquad i = 1, 2 \tag{28}$$

Once again analysis of the last three equations will show that if the contributions from $T_{ij}C_j$ ($i = 2, 4, 5$; $j = 2, 3$) are not to be of order of magnitudes greater than the forcing terms we must have $C_j' \leq O[\Omega^{-(2np-7+3p)/2}]$; $j = 2, 3$, which upon substitution into the first two equations shows that the contribution from each of the higher modes has an upper limit of $O[\Omega^{-np+5-3p}]$. The ratio of this limit to any of the remaining terms in the first two equations is of $O(\Omega^{3-p})$. Therefore, if we restrict ourselves to those terms with the largest order of magnitudes, the higher modes will not contribute to the solution. This time, however, it is apparent from the form of these largest order of magnitude terms that, to this level of approximation, the first two equations are not linearly independent. To see which of the infinity of pairs of unknowns C_1 and C_4 that satisfy these dependent equations is the continuous extension of the limiting process requires our taking the next term in the expansions of T_{i1} and T_{i4} ($i = 1, 2$) in powers of Ω. The ratio of the second term to the first in each of these expansions is $O(\Omega^2)$, provided $n \neq 1$ in which case it is $O(\Omega^2 \ln \Omega)$. In either event, it is seen that the second terms represent a larger order of magnitude contribution to the first two equations than do the higher modes as long as $p < 1$, that is, as long as S goes to infinity as Ω goes to zero. It is not difficult to show that carrying out the expansions to two terms is sufficient to obtain two linearly independent equations.

Case 3. In the third case, it is desired to keep S finite as Ω approaches zero. This case is similar to Case 2, if p is set equal to one, with the exception that the asymptotic expansions of the Hankel functions of arguments Z_2S, Z_3S and Z_4S are no longer valid. Therefore, it is not possible to simplify any of the expressions T_{ij} ($i = 1, \ldots, 5$; $j = 2, 3$) or T_{i5} ($i = 3, 4, 5$). It also is apparent that each of these expressions are of $O(1)$. Proceding as before, the last three equations limit C_j ($j = 2, 3$) to be less than $O(\Omega^{-n+4})$; hence, the maximum contributions from the higher modes to the first two equations are all of $O(\Omega^{-n+4})$. These contributions are of $O(\Omega^2)$ compared to the remaining terms of these two equations if all expansions in powers of Ω are carried out to a single term. As in Case 2, however, truncation of the expansions of T_{ij} ($i = 1, 2$; $j = 1, 4$) after a single term results in a pair of dependent equations. This necessitates our taking two terms of the expansion if we are to uniquely determine C_1 and C_4. The second term of each of these expansions is of $O(\Omega^2)$ relative to the first, provided $n \neq 1$, which indicates that their contributions to the first two equations are the same order of magnitudes as those from the higher modes. It is therefore not justifiable to retain them while not retaining the terms arising from the higher order modes. This leads to the conclusion that if $S = O(1)$ the second order solution will not "reduce" to the generalized plane stress solution in the limit as Ω approaches zero.

The case of $n = 1$ cannot be included in the above conclusion since in this case the second terms of the expansions in question are $O(\Omega^2 \ln \Omega)$ relative to the first and hence these terms offer a contribution to the first two equations which is greater than that of the higher order modes.

Case 4. In the final case, which is introduced for completeness, S is allowed to approach zero as Ω approaches zero. This case is also similar to Case 2, if the range of p is greater than one, with the exception that the Hankel functions of arguments Z_2S, Z_3S and Z_5S are now expanded as small argument expansions. If this is done the forms of T_{ij} ($i = 1, \ldots, 5$; $j = 2, 3$) are the same as T_{i1} ($i = 1, \ldots, 5$) given in equations (27) with Z_j ($j = 2, 3$) replacing

Z_1. Furthermore, the magnitude of T_{ij} $(i = 1, \ldots, 5;\ j = 2, 3)$ relative to Ω is given by

$$\begin{aligned} &T_{ij}\ (i = 1, \ldots, 5;\ j = 2, 3) \text{ are } O[\Omega^{-(n+2)(p-1)}] \\ &T_{5j}\ (j = 2, 3) \text{ are } O[{}^{-(n+1)(p-1)}] \end{aligned} \tag{29}$$

Once again, the last three equations require $C_j\,(j = 2, 3) \leq O(\Omega^{-n+2})$, so that the maximum contribution of the higher modes to the first two equations is $O(\Omega^2)$ compared to the remaining terms. The comments regarding the linear dependence of the first two equations if only one term in the expansions is retained are still valid and, therefore, the same conclusions can be drawn as were drawn for the case of S is $O(1)$.

ADDITIONAL SOLUTIONS

One additional limiting form of the solution that is of interest is the case in which S approaches zero while Ω is retained as a finite parameter. Such a limit could be interpreted physically to correspond to the case of the reflection of an infinite train of plane waves from the traction free edge of a semi-infinite plate. This latter problem was solved within the framework of the Mindlin–Medick equations by Gazis and Mindlin [5]. Expansion of the solution given in equations (15) in terms of powers of S and retention of only the first term will result in the solution given by them. It is, therefore, not unreasonable to expect that certain aspects of their solution will carry over to the present problem. Of particular interest is the possibility of a "resonance" like phenomenon being set up in the vicinity of the boundary for $\Omega = 1{\cdot}314$.

As a final case the mixed boundary value problem could be considered. In this case the three boundary conditions

$$\tau_{r\theta}^{(0)}\,(r = a) = \tau_{2r}^{(1)}\,(r = a) = \tau_{r\theta}^{(2)}\,(r = a) = 0 \tag{30}$$

are replaced by the conditions

$$u_\theta^{(0)}\,(r = a) = u_2^{(1)}\,(r = a) = u_\theta^{(2)}\,(r = a) = 0 \tag{31}$$

Using these conditions the system of equations which give the reflected amplitudes has the same form as before [i.e., equations (15)], however, now the second, fourth and fifth row become

$$\begin{aligned} &T_{2j} = \frac{n}{S} H_n^{(1)}(Z_j S) \qquad j = 1, 2, 3 \\ &T_{4j} = \beta_j T_{2j} \qquad j = 1, 2, 3 \\ &T_{5j} = \bar{\alpha}_j H_n^{(1)}(Z_j S) \qquad j = 1, 2, 3 \\ &T_{24} = \frac{n}{S} H_n^{(1)}(Z_4 S) - Z_4 H_{n+1}^{(1)}(Z_4 S) \\ &T_{45} = \frac{n}{S} H_n^{(1)}(Z_5 S) - Z_5 H_{n+1}^{(1)}(Z_5 S) \\ &T_{55} = 0 \end{aligned} \tag{32}$$

An asymptotic analysis similar to that accomplished for the traction free boundary would indicate somewhat similar results with the exception that the first two equations in the mixed case are always independent when only one term of the expansion in Ω is retained. For this reason the solutions predicted by the generalized plane stress theory and the second order equations are identical in the limit of Ω approaching zero, independent of the size of the S, if the boundary conditions are of the mixed type.

CONCLUSIONS

In conclusion, then, the problem considered was the reflection of a wave, propagating in the lowest extensional mode, from a traction free circular cavity. The solution was obtained within the framework of both the generalized plane stress theory and that of a second order theory. The amplitudes of the reflected waves that are contained in the two theories were then compared to see under what conditions they could be brought into analytical agreement. The results have shown that the two solutions will agree in the limit of the vanishing of the non-dimensionalized frequency thickness product as long as the ratio of hole diameter to plate thickness goes to infinity.

The question could be raised as to how different "numerical" results would be for various ranges of S and Ω. It would be difficult to justify the rather involved numerical program that would be needed to obtain such results, however, since there appears to be no way to correlate how different the two numerical results are from one another to how different either of them is from the "true" value. Of greater interest, from a numerical viewpoint, is the possible existence of a resonant-like vibration confined to the vicinity of the cavity. Such a vibration was found by Gazis and Mindlin for the case of a plane boundary and, as pointed out by them, the prediction of such a phenomenon is beyond the framework of the generalized plane stress theory. The undertaking of such a numerical program is being contemplated.

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Résumé—L'emploi est fait d'une théorie approximative d'une plaque à deux dimensions, qui prend en considération quelques aspects de l'épaisseur de plaque pour étudier la réflection d'un train infini d'ondes extensionnelles d'une limite circulaire libre de traction. Les résultats sont investigués afin de voir sous quelles conditions ces résultats pourront être en accord analytique avec ceux prèdits par la théorie de tension plane generalisée lorsque cette dernière est employée pour résoudre le même problème.

Zusammenfassung—Eine annähernde zwei dimensionale Plattentheorie, die die Plattendicke teilweise berücksichtigt wird angewandt um Reflektionen einer unendlichen Reihe von Streckungswellen von einer kreisförmigen spannungsfreien Grenzfläche zu untersuchen. Die Resultate werden genau untersucht, um festzustellen unter welchen Umständen sie analytische Übereinstimmung mit den Resultaten zeigen, die vorausgesagt wurden entsprechend der verallgemeinerten Theorie des ebenen Spannungszustandes, falls diese zur Lösung desselben Problemes versucht wird.

Абстракт—Применяется приблизительная теория двумерных пластин, которая принимает во внимание некоторые аспекты толщины пластины для изучения отражения бесконечного ряда экстенсиональных волн от свободной круговой границы. Результаты исследуются, чтобы узнать при каких условиях их можно привести к аналитическому соглашению с результатами, предсказанными теорией обобщённого плоского напряжения, когда последняя применяется для решения той же проблемы.